The Two Envelope Paradox

A beggar is walking past a temple during a religious festival when a monk motions for the beggar to enter an ornate alcove. The monk, smiling broadly, says

> **"Brother, fate has chosen to reward you. You may chose either of these two envelopes. Both contain money, but one contains twice as much money as the other. I don't know which is which."**

The beggar feels both envelopes and they both seem the same. He chooses one of the envelopes and turns it over to open it. There is writing on the back but the beggar can't read so he just opens the envelope, finds the money, and runs out to buy some food, barely pausing to thank the still smiling monk.

The next year, during the same religious festival, a maths student is walking past the temple. The monk, smiling broadly, gives the same speech as before. The maths student weighs up the envelopes and can find no difference in mass, sound when shaken, compressibility, or any other test he can think of. Reluctantly he picks one envelope and turns it over to open it. On the back there is writing.

> **"The other envelope contains either double the amount in this envelope, or half the amount in this envelope."**

The maths student is elated. He realises that his years of study have finally paid off. If the amount in this envelope is M then, since either possibility is equally likely, the expectation when choosing the other envelope is $2M \times \frac{1}{2} + \frac{1}{2}M \times \frac{1}{2} = \frac{5}{4}M$. By swapping envelopes he gets 25% more money on average. He swaps envelopes, turns the new envelope over and it has exactly the same writing on the back. He understands the problem now so he swaps the envelopes again, and again, and again, … and then his smile begins to fade. Something is not right here! He looks to the monk for inspiration but the monk just continues to smile, and makes a two-handed palm-up gesture to the effect that he doesn't know. The maths student is stuck in a paradox.

If you do not understand the idea of expectation value then you will not understand the math student's paradox. In this case there is no point in reading on because the answer will not be understandable either.

The interesting thing about this problem is that the uninitiated and the expert come up with the same answer. It is only the half-educated student that suffers from confusion!

Try to resolve the paradox in you own mind before reading the solution overleaf.

No cheating and no peeking!

The Two Envelope Paradox - Resolution

There have been articles published on this paradox and you can search the internet to find at least some of them. Some solutions include opening the envelope and using considerations on the possible values of money in the envelopes. These miss the beauty and simplicity of the paradox.

To the layman the problem is straightforward. Both envelopes are equally likely to contain the larger sum of money. There is nothing to distinguish between them so any guess is as good as any other guess. Swapping envelopes is pointless.

The maths student is *apparently* using a good theory of expectation value. Consider playing a game where you bet £100 on the toss of a fair coin. In this game a win gains you £90, plus your £100 returned to you, whereas a lose costs you the £100. You might reasonably feel the "odds were not on your side" in this game, and you would be right! The expectation value is $£90 \times \frac{1}{2} - £100 \times \frac{1}{2} = -£5$. In other words, on average, every time you make that bet you lose $£5$.

Suppose the two envelopes contain S and $2S$. On the first pick of the envelope the expectation value is therefore $S \times \frac{1}{2} + 2S \times \frac{1}{2} = \frac{3}{2}S$. The student feels that by swapping envelopes he attains a new value $2M \times \frac{1}{2} + \frac{1}{2}M \times \frac{1}{2} = \frac{5}{4}M$. Putting $M = \frac{3}{2}S$ tells him that by swapping envelopes he should increase the value to $\frac{15}{8}S$. That is clearly absurd since he has no way of differentiating between the envelopes. It becomes even more absurd when he swaps them again and calculates the expectation value as $\frac{75}{32}S$ since the average win now exceeds the larger sum of money in the envelopes!

I have seen this paradox explained as a subtle manipulation of the distribution of random numbers, but it is no such thing. In actual fact this is a straight forward mis-application of conditional probability theory. The formula, $2M \times \frac{1}{2} + \frac{1}{2}M \times \frac{1}{2} = \frac{5}{4}M$, assumes that the second choice is *independent* of the first choice. This is the error and this is the source of the apparent paradox.

If the chosen envelope contains S we guarantee to swap it for one containing 2S, a loss of S. If the chosen envelope contains 2S we guarantee to swap it for one containing S, a gain of S. The expectation value of a swap is therefore zero since we gain S with probability $\frac{1}{2}$ in the first case, and we lose S with probability $\frac{1}{2}$ in the second case. The expectation value was correctly assessed in the first instance as $S \times \frac{1}{2} + 2S \times \frac{1}{2} = \frac{3}{2}S$, but since S is unknown that really doesn't tell us much.

I came across this paradox on the Dr Math forum, but I found the answer and the linked answers rather unsatisfactory. I have embellished the story line of the paradox whilst retaining the core mathematical problem.